THE SINGULAR EIGENFUNCTION METHOD: THE MILNE PROBLEM FOR ISOTROPIC AND EXTREMELY ANISOTROPIC SCATTERING

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ABSTRACT:

In the \( C_N \) method of solving the third form of the transport equation, the medium as a result of Placzek lemma is extend to infinity. Infinite medium Green function which is obtained by the Fourier transform technique is used and the method is applied to one velocity problems in plane and cylindrical geometries. As the result of physical applications of the \( C_N \) method in different geometries, it is seen that the only difficulty lies in writing the expression of the Green’s function in a form easy to handle. In the new method of solving of the third form of the transport equation (that we have generated recently), three methods, namely, \( C_N \), \( F_N \), and the method of elementary solutions are considered, compared and the Green function in terms of the singular eigenfunctions is used. This method yields simple analytical expressions that can be solved numerically more efficiently than the \( C_N \) method because the expression of the Green function is in the form easy to handle. Here this new method is applied to calculate the extrapolation length for the Milne problem which is classical problem in astrophysics concerned with the diffusion of radiation through a stellar atmosphere for both isotropic and extremely anisotropic scatterings. It is shown that the numerical results which are tabulated for selected cases are accurate even in the approximations of the lowest order and are in good agreement with the numerical results obtained by the other methods.

I. INTRODUCTION

Linear transport equations arise in the fields of radiative transfer, neutron diffusion, the theory of plasmas and the theory of the sound propagation. In neutron transport theory, the neutron transport equation is employed to solve to some physical problems as albedo, criticality, extrapolation lengths for Milne problem for isotropic and anisotropic scattering cases. There are three forms of neutron transport equation. The first form which is also known as Boltzmann equation is solved by using the methods of Case, \( F_N \), \( P_L \), the second form is solved by using variational method and the third form of the neutron transport equation is solved by using \( C_N \) method.

The \( C_N \) method is based on Placzek lemma which is converted the finite medium into the infinite medium. An integral equation is provided for the angular flux at the boundaries of the various media which allows the fluxes to be calculated at any point \([1,2]\). For plane geometry \( C_N \) equations can be written as
\[ 0 = \int_{1}^{0} G(\mu, \mu') \nu^{+}(\mu') d\mu' + \int_{-1}^{0} G(\mu, \mu') \nu^{-}(\mu') d\mu', \quad \mu \geq 0 \tag{1} \]

\[ \nu^{-}(\mu) = \int_{0}^{1} G(\mu, \mu') \nu^{+}(\mu') d\mu' + \int_{-1}^{0} G(\mu, \mu') \nu^{-}(\mu') d\mu', \quad \mu < 0 \tag{2} \]

where \( G \) is the infinite medium Green’s function and \( G(x, \mu, \mu') \) is angular flux at \( x \) in the direction \( \mu \) produced by a plane unit source at the origin, emitting in the direction \( \mu' \). It is obtained from Fourier transform technique as.

\[ G(x, \mu, \mu') = \frac{e}{4\pi} \int_{-\infty}^{\infty} \frac{\exp(-ikx)dk}{(1-ik\mu)(1-ik\mu') \left( 1 - \frac{e}{\tan^{-1}k} \right)} \tag{3} \]

In the new method of solving the third form of the transport equation that we have generated recently, the infinite medium Green’s function which is obtained from the solution of the Boltzmann equation is used to solve \( C_n \) equations [3]. The Green’s function \( G(x_0 \rightarrow x; \mu_0 \rightarrow \mu) \) represents the angular flux at \( x, \mu \) due to a unit plane source at \( x_0, \mu_0 \). For isotropic scattering, [4,5]

\[ \frac{\partial G(x_0 \rightarrow x; \mu_0 \rightarrow \mu)}{\partial x} + G(x_0 \rightarrow x; \mu_0 \rightarrow \mu) = \frac{c}{2} \int_{-1}^{1} G(x_0 \rightarrow x; \mu_0 \rightarrow \mu') d\mu' + \delta(x-x_0)\delta(\mu-\mu_0) \tag{4} \]

\[ G^{\pm}(x_0 \rightarrow x; \mu_0 \rightarrow \mu) = \frac{\phi(\pm \nu_0, \mu) \phi(\pm \nu_0, \mu)}{N(\nu_0)} e^{\frac{|x-x_0|}{\nu_0}} + \int_{0}^{1} \frac{\phi(\pm \nu, \mu) \phi(\pm \nu, \mu)}{N(\nu)} e^{\frac{|x-x_0|}{\nu}} d\nu \tag{5} \]

where the upper sign apply for \( x < x_0 \), the lower for \( x > x_0 \) and

\[ \phi(\pm \nu_0, \mu) = \pm \frac{c \nu_0}{\nu - \mu}, \quad \phi(\nu, \mu) = \frac{1}{\nu - \mu} \]

\[ \phi(\nu, \mu) = \frac{c \nu}{\nu - \mu} + \lambda(\nu) \delta(\nu - \mu), \quad \lambda(\nu) = 1 - c \nu \tanh^{-1} \nu \tag{6} \]

are known as the discrete and continuous singular eigenfunctions corresponding to the discrete and continuum eigenvalues \( \nu_0 \) and \( \nu \) respectively.
\[ N(\pm v_0) = \pm \frac{cv_0^3}{2} \left[ \frac{e}{v_0^2 - 1} - \frac{1}{v_0^2} \right] \] \quad (8)

\[ N(v) = \sqrt{\left(1 - c v \tanh^{-1} v\right)^2 + \frac{e^2 \pi^2 v^2}{4}} \] \quad (9)

correspond to the normalization constants of the discrete and continuum modes.

II. MILNE PROBLEM

Let us consider the half-space x > 0, there is a source at infinity and vacuum at x< 0. To solve this problem using Placzek lemma, a negative source is created at x=0 and the medium is extended to infinity. The angular flux at the origin from a source at infinity is proportional to \( \frac{v_0}{v_0 + \mu} \). The contribution of the source to the angular flux at the surface can be written in terms of the infinite medium Green’s function. Since there is no incoming current at x=0, the net angular flux in the plane geometry is given by [1,2]

\[ 0 = \int_{-1}^{0} G(\mu, \mu') v^- (\mu') \mu' d\mu' + \frac{v_0}{v_0 + \mu} \quad \mu > 0 \] \quad (10)

\[ v^- (\mu) = \int_{-1}^{0} G(\mu, \mu') v^- (\mu') \mu' d\mu' + \frac{v_0}{v_0 + \mu} \quad \mu < 0 \] \quad (11)

where outgoing angular flux is defined by

\[ v^- (\mu) = \sum_{l=0}^{N} \mu^l v^-_l \] \quad (12)

and is obtained by substituting it into Eqs. (10) and (11) and solving for the coefficients \( v^-_l \). To find the extrapolation length and the asymptotic flux is written as a summation of the asymptotic parts both from the source at infinity and the source at the surface.

\[ \phi_{as} (x) = v_0 e^{v_0} \ln \left( \frac{v_0 + 1}{v_0 - 1} \right) + \int_{-1}^{1} d\mu \int_{-1}^{0} G(x, \mu, \mu') v^- (\mu') \mu' d\mu' \] \quad (13)

III. THE HALF-SPACE EXTRAPOLATION LENGTH FOR ISOTROPIC SCATTERING

Writing Eqs. (10) and (11) as

\[ 0 = \int_{-1}^{0} G^- (\mu, \mu') v^- (\mu') \mu' d\mu' + \frac{v_0}{v_0 + \mu} \quad \mu > 0 \] \quad (14)
\[ v^{-} (\mu) = \int_{-1}^{0} G^+ (\mu, \mu') v^{-} (\mu') \mu' \, d\mu' + \frac{v_0}{v_0 + \mu}, \quad \mu < 0 \]  

(15)

using the infinite medium Green's function given in Eq. (5) for isotropic scattering case and multiplying these equation by \( \mu^m \) and integrating over \( \mu \in [(0,1), (-1,0)] \), respectively, we obtain

\[
\sum_{l=0}^{N} v_l^{-} (-1)^l \left\{ \frac{e^2 v_0^2}{4 N(v_0)} B_l(v_0) A_{m-1}(v_0) + \frac{e^2}{4} I_{l,m-1}^{AB} \right\} = v_0 A_{m-1}(v_0) \quad ,(16)
\]

\[
\sum_{l=0}^{N} v_l^{-} (-1)^l \left\{ \frac{e^2 v_0^2}{4 N(v_0)} A_l(v_0) A_{m-1}(v_0) + \frac{e^2}{4} I_{l,m-1}^{AB} + \frac{1}{l+m+1} \right\} = v_0 B_{m-1}(v_0) \quad ,(17)
\]

where

\[ A_l(v_0) = \frac{2}{c v_0} \int_0^1 \mu^{l+1} \phi(-v_0, \mu) d\mu \quad , \quad B_l(v_0) = \frac{2}{c v_0} \int_0^1 \mu^{l+1} \phi(v_0, \mu) d\mu \quad (18) \]

and

\[
A_0(v_0) = \frac{1}{l+1} - v_0 A_{l-1}(v_0), \quad \quad B_0(v_0) = \frac{2}{c} - 1 - v_0 \log \left(1 + \frac{1}{v_0}\right) \quad (19)
\]

\[
I_{l,m-1}^{AB} = \int_0^1 \frac{v^2 A_l(v) A_{m-1}(v)}{N(v)} \, dv \quad , \quad I_{l,m-1}^{AB} = \int_0^1 \frac{v^2 A_l(v) B_{m-1}(v)}{N(v)} \, dv \quad (20)
\]

Similarly replacing \( G \) in Eq.(13) by \( G^- \) which is given in Eq.(5) and using Eq.(12) we get

\[
\phi_{in}(x) = v_0 e^{v_0} \ln \left( \frac{v_0 + 1}{v_0 - 1} \right) + \frac{c v_0 e^{v_0}}{N(v_0)} \sum_{l=0}^{N} v_l^{-} (-1)^l + 1 A_l(v_0) \quad (21)
\]

The extrapolated asymptotic flux i.e. the flux extended by its natural curvature with distance will vanish when \( x = -z_0 \) and the distance \( z_0 \) is called the extrapolation length.

\[
z_0 = -\frac{v_0}{2} \ln \left( \frac{\phi_N (v_0)}{\phi_{\infty} (v_0)} \right) \quad (23)
\]

where

\[
\phi_{\infty} (v_0) = v_0 \ln \left( \frac{v_0 + 1}{v_0 - 1} \right) = \frac{2}{c} \quad (24)
\]
The infinite medium Green’s function for extremely anisotropic scattering satisfies the following equation

\[
\left\{ \mu \frac{d}{dx} + 1 - \frac{c}{2} \right\} f(\mu, \mu_0) d\mu_0 \right\} G(x_0 \rightarrow x; \mu_0 \rightarrow \mu) = \delta(x - x_0) \delta(\mu - \mu_0) \]  

(26)

where

\[
f(\mu, \mu_0) = \frac{a}{2} + b \delta(\mu + \mu_0) + d \delta(\mu - \mu_0)
\]

and its solution can be expressed in terms of Green’s functions for isotropic scattering as [6]

\[
G_{\text{anis}}(x_0 \rightarrow x; \mu_0 \rightarrow \mu) = \frac{p + q}{2q} G^{s, c'}(x_0 \rightarrow x'; \mu_0 \rightarrow \mu) + \frac{d}{2q} G^{s, c'}(x_0 \rightarrow x'; -\mu_0 \rightarrow -\mu)
\]

(27)

where

\[x' = xq, \quad \nu_0' = q\nu_0, \quad p = 1 - bc, \quad q = \left(\frac{p^2 - d^2 c^2}{2c}\right)^{\frac{1}{2}}, \quad c' = \frac{ac}{1 - bc - dc}.
\]

Eqs. (10) and (11) for extremely anisotropic scattering become

\[
0 = \int_{-1}^{0} G^{-\text{anis}}(\mu, \mu') v^{-}(\mu') \mu' d\mu' + \nu_0 \left( \frac{p + q + dc}{\nu_0 + \mu} + \frac{p - q + dc}{\nu_0 - \mu} \right) \quad \mu > 0
\]

(29)

\[
v^{-}(\mu) = \int_{-1}^{0} G^{+\text{anis}}(\mu, \mu') v^{-}(\mu') \mu' d\mu' + \nu_0 \left( \frac{p + q + dc}{\nu_0 + \mu} + \frac{p - q + dc}{\nu_0 - \mu} \right) \quad \mu < 0
\]

(30)

Multiplying Eq. (29) by \(\mu'^m\) and integrating over \(\mu \in (0,1)\), we obtain
Using the definition

$$\phi^{an, os}(x) = \frac{1}{c} \frac{p+dc}{q} \nu_0^{N} \sum_{l=0}^{N} V_l (-1)^{l+1} \begin{bmatrix} \left( p + q \right) A_l (v_0') + dc A_l (v_0') \\ + dc B_l (v_0') + (p - q) B_l (v_0') \end{bmatrix}$$

(32)

where

$$\phi_\infty (v_0') = \frac{c \nu_0^N}{a q N (\nu_0)} \sum_{l=0}^{N} V_l (-1)^{l+1} \begin{bmatrix} \left( p + q \right) A_l (v_0') + dc A_l (v_0') \\ + dc B_l (v_0') + (p - q) B_l (v_0') \end{bmatrix}$$

(34)

$$z_0 = -\frac{\nu_0}{2q} \log \left( -\frac{\phi_\infty}{\phi_\infty} \right)$$

(35)

the extrapolation length is calculated for isotropic and extremely anisotropic scatterings for both $c>1$ and $c<1$ by solving $C_N$ equations using the singular eigenfunctions of the method of elementary solutions. If $c>1$, $\nu_0$ in Eqs. (29), (30), (32) should be replaced by $-i\nu_0$.

**CONCLUSIONS**

This new method leads to simple expressions for solving numerically. Our numerical results are also accurate even in the approximations of the lowest order as in the $C_N$ method. Numerical results for both $c<1$ and $c>1$ which are tabulated for some selected cases are in agreement with those obtained by the other methods [5,7,8].

**REFERENCES**


**Table I.** Values of $c_\alpha z_0$ for isotropic scattering

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**Table II.** Values of $c_\alpha z_0$ for forward scattering

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**Table III.** Values of $c_\alpha z_0$ for backward scattering

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