ON A PROBLEM WITH EXTREMELY ANISOTROPIC SCATTERING IN SLAB

A. Kaşkaş, M. Karahasanıoğlu
Ankara University, Faculty of Science, Department of Physics 06100 Tandogan, Ankara Turkey

The slab albedo problem will be solved for the extremely anisotropic scattering function using the analytical expressions of \( F \) method with a different numerical approximation. The infinite medium Green’s function for the anisotropic scattering obtained in terms of the infinite medium Green’s function for the isotropic scattering is used. The albedo and transmission factor for a slab are calculated and the numerical results are compared with data available in the literature.

1. INTRODUCTION

The \( F \) method was used to solve some problems in neutron transport theory such as half-space albedo problem, the half-space constant source problem, slab albedo problem, the critical slab problem etc. [1-3]. This method was developed from the use of Placzek lemma as \( C \) method [5-7]. The integro-differential form of the one-speed neutron transport equation in plane geometry is given by

\[
\mu \frac{\partial \Psi (x, \mu)}{\partial x} + \Psi (x, \mu) = c \int_{-1}^{1} \Psi (x, \mu) f (\mu, \mu') d \mu' + s(x, \mu)
\]

(1)

Here \( \Psi (x, \mu) \) is the neutron angular distribution at any point, \( s(x, \mu) \) is the source function, \( c \) is the mean number of secondary neutrons per collision, \( \mu \) is the direction cosine of the propagating neutrons, \( f (\mu, \mu') \) is the scattering function and normalized to unity. The infinite medium problem for the half-space is defined by [2,3]

\[
\mu \frac{\partial \Psi_1 (x, \mu)}{\partial x} + \Psi_1 (x, \mu) = c \int_{-1}^{1} \Psi_1 (x, \mu') f (\mu, \mu') d \mu' + s(x, \mu) H(x) + \mu \Psi (0, \mu) \delta (x)
\]

(2)

Where \( \mu \Psi (0, \mu) \delta (x) \) represents the sources on the surface at \( x = 0 \) and \( H(x) \) is a unit step function which is defined as

\[
H(x) = \begin{cases} 
1 & \text{for } x \in (0, \infty) \\
0 & \text{otherwise}
\end{cases}
\]

(3)

\( \Psi_1 (x, \mu) = H(x) \Psi (x, \mu) \)

To write an analytical expression for \( \Psi_1 (x, \mu) \), we need to define the infinite medium Green’s function. \( G(x_0 \to x; \mu_0 \to \mu) \) represents the infinite medium Green’s function which satisfies the following equation [4]

\[
\frac{\partial G(x_0 \to x; \mu_0 \to \mu)}{\partial x} + G(x_0 \to x; \mu_0 \to \mu) = c \int_{-1}^{1} G(x_0 \to x; \mu_0 \to \mu') f (\mu, \mu') d \mu' + \delta (x - x_0) \delta (\mu - \mu_0)
\]

(4)

subject to

\( G(x_0 \to \pm \infty; \mu_0 \to \mu) = 0 \)

(5)
It means that we seek the distribution from a plane source of unit strength at \( x = x_0 \) emitting in the direction \( \mu = \mu_0 \). \( G(x_0 \rightarrow x; \mu_0 \rightarrow \mu) \) is the angular flux at \( x \) in the direction \( \mu \) in an infinite homogeneous medium, produced by a plane unit source at \( x_0 \) emitting in the direction \( \mu_0 \). Then for the half-space problem, \( \Psi_1(x, \mu) \) can be written as [2,3]

\[
\Psi_1(x, \mu) = \int_{-\infty}^{\infty} d\mu_0 \int_{x_0}^{\infty} dx_0 \psi(x_0 \rightarrow x; \mu_0 \rightarrow \mu) \left[ H(x_0)\psi(x_0, \mu_0) + \mu_0 \Psi(0, \mu_0)\delta(x_0) \right].
\]  

(6)

In the extremely anisotropic scattering case, \( f(\mu, \mu') \) is defined by

\[
f(\mu, \mu') = \frac{a}{2} + b\delta(\mu - \mu') + d\delta(\mu + \mu').
\]  

(7)

\( a = 1 - \alpha, \ b = \alpha, \ d = 0 \) represents forward scattering with isotropic scattering, \( a = 1 - \alpha, \ b = 0, \ d = \alpha \) represents backward scattering with isotropic scattering, \( \alpha = 0 \) corresponds to the isotropic scattering. The infinite medium Green’s function for anisotropic scattering case has been obtained in terms of the infinite medium Green’s function for the isotropic scattering as [8]

\[
G_{\text{anis}}(x_0 \rightarrow x; \mu_0 \rightarrow \mu) = \frac{p + q}{2q} G_{\text{is}, c', c}(x_0 \rightarrow x'; \mu_0 \rightarrow \mu) + \frac{dc}{2q} G_{\text{is}, c', c}(x_0 \rightarrow x'; \mu_0 \rightarrow -\mu) + \frac{dc}{2q} G_{\text{is}, c', c}(x_0 \rightarrow x'; \mu_0 \rightarrow -\mu)
\]  

(8)

in Eq. (8), \( G_{\text{is}, c', c} \) is replaced by

\[
G_{\text{is}, c', c}(x_0 \rightarrow x'; \mu_0 \rightarrow \mu) = \frac{\phi'(\pm v'_0, \mu) \phi'(\pm v'_0, \mu_0)}{N(v'_0)} \exp \left( \frac{|x' - x_0'|}{v'_0} \right)
\]  

(9)

\[
+ \int_0^1 \frac{\phi'(-v', \mu) \phi'(v', \mu_0)}{N(v')} \exp \left( \frac{|x' - x_0'|}{v'} \right) dv',
\]

where the upper sign is for \( x < x_0 \), the lower sign is for \( x > x_0 \) and \( x' = xq \), \( v'_0 = qv_0 \), \( p = 1 - bc \), \( q = \sqrt{p^2 - d^2c^2} \), \( c' = \frac{ac}{1 - bc - dc} \).

\( \pm v'_0 \) are the two discrete eigenvalues with eigenfunctions \( \phi'(\pm v'_0, \mu) \) and \( \pm v' \) are the continuum eigenvalues on the range \([-1, 1]\) with eigenfunctions \( \phi'(-v', \mu) \). \( \phi'(\pm v', \mu) = \pm \frac{c'v'_0}{2} \frac{1}{\pm v'_0 - \mu} \)

(11)

with

\[
\lambda(v') = 1 - c'v'\tanh^{-1}v'.
\]  

(12)

The normalization constants corresponding to discrete and continuum modes are respectively

\[
N'(\pm v'_0) = \int_{-1}^{1} \mu \phi'(\pm v'_0) \phi'(\pm v'_0) d\mu = \pm \frac{c'v'^3}{2} \left[ \frac{c'}{v'^2} - \frac{1}{v'^2} \right]
\]

(13)

\[
N'(v') = \int_{-1}^{1} \mu \phi'(\pm v') \phi'(\pm v') d\mu = v' \left\{ \left( 1 - c'v'\tanh^{-1}v' \right)^2 + \frac{c'^2\pi^2v'^2}{4} \right\}
\]
2. SLAB ALBEDO PROBLEM

We consider the finite slab of thickness $2\tau (-\tau \leq x \leq \tau)$, [2,3,5,6,10-16]. In $F_N$ method, the infinite medium problem for a finite slab is defined by

$$\mu \frac{\partial \Psi_1(x, \mu)}{\partial x} + \Psi_1(x, \mu) = c \int_{-\mu}^{+\mu} \Psi_1(x, \mu') f(\mu, \mu') d\mu' + s(x, \mu) H^*(x)$$

$$\quad + \mu' \Psi(x, \mu) \left[ \delta(x + \tau) - \delta(x - \tau) \right] .$$

(14)

Where the last term of Eq.(14) represents the sources on the surface at $x = \tau$ and $x = -\tau$. $H^*(x)$ is a unit step function

$$H^*(x) = \begin{cases} 1 & \text{for } x \in [-\tau, \tau] \\ 0 & \text{otherwise} \end{cases}$$

(15)

$$\Psi_1(x, \mu) = H^*(x) \Psi(x, \mu)$$

$$\Psi_1(x, \mu) = \frac{1}{2} d\mu_0 \left\{ \int_{-\mu_0}^{+\mu_0} dx_0 G(x_0 \rightarrow x; \mu_0 \rightarrow \mu) s(x_0, \mu_0) + G(-r \rightarrow x; \mu_0 \rightarrow \mu) \mu_0 \Psi(-r, \mu_0) \right\} - G(r \rightarrow x; \mu_0 \rightarrow \mu) \mu_0 \Psi(r, \mu_0) .$$

(16)

The exit distributions at $x = \pm \tau$ are given by [2,3]

$$\Psi(-\tau, -\mu) = -\int_{-\mu}^{+\mu} d\mu_0 G(-\tau \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(-\tau, -\mu_0)$$

$$- \int_{-\mu}^{+\mu} d\mu_0 G(-\tau \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(-\tau, -\mu_0) + \int_{-\mu}^{+\mu} d\mu_0 G(r \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(r, -\mu_0) - \int_{-\mu}^{+\mu} d\mu_0 G(r \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(r, -\mu_0) .$$

(17a)

$$\Psi(\tau, \mu) = -\int_{-\mu}^{+\mu} d\mu_0 G(-\tau \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(-\tau, -\mu_0)$$

$$+ \int_{-\mu}^{+\mu} d\mu_0 G(-\tau \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(-\tau, -\mu_0) + \int_{-\mu}^{+\mu} d\mu_0 G(r \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(r, -\mu_0) - \int_{-\mu}^{+\mu} d\mu_0 G(r \rightarrow -\tau^*; -\mu_0 \rightarrow -\mu) \mu_0 \Psi(r, -\mu_0) .$$

(17b)

with

$$s(x, \mu) = 0$$

(18)

$$\Psi(-\tau, \mu) = \mu^\beta$$

$$\Psi(\tau, -\mu) = 0$$

(19a)

$$\Psi(-\tau, -\mu) = \sum_{i=0}^{N} a_i \mu^i$$

$$\Psi(\tau, \mu) = \sum_{i=0}^{N} b_i \mu^i$$

(19b)
Multiplying the exit distributions in Eqs.(17) by \( \mu^{n+1} \) and integrating over \( \mu \in [0, 1] \), using Eqs.(19) similar to \( H_n \) method, we obtain [8]

\[
\sum_{i=0}^{N} a_i D_{m \ell} + \sum_{i=0}^{N} b_i E_{m \ell} = F_{m \beta} \tag{20a}
\]

\[
\sum_{i=0}^{N} a_i E_{m \ell} + \sum_{i=0}^{N} b_i D_{m \ell} = G_{m \beta} \tag{20b}
\]

\[
D_{m \ell} = \frac{2q}{m + \ell + 2} + (p + q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_m(v_0') A_{m}(v_0) + \frac{c_{2}^{2}}{4} I_{m\ell}^{AA} \right) + dc \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} B_m(v_0') A_{m}(v_0) + \frac{c_{2}^{2}}{4} I_{m\ell}^{AB} \right) + (p - q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} B_m(v_0') B_{m}(v_0) + \frac{c_{2}^{2}}{4} I_{m\ell}^{BB} \right) \tag{21a}
\]

\[
E_{m \ell} = (p + q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_m(v_0') B_{m}(v_0) \exp\left( \frac{-2 \tau'}{v_0} \right) + \frac{c_{2}^{2}}{4} I_{m\ell}^{ABE} \right) + dc \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_m(v_0') B_{m}(v_0) \exp\left( \frac{-2 \tau'}{v_0} \right) + \frac{c_{2}^{2}}{4} I_{m\ell}^{ABE} \right) + (p - q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_m(v_0') B_{m}(v_0) \exp\left( \frac{-2 \tau'}{v_0} \right) + \frac{c_{2}^{2}}{4} I_{m\ell}^{ABE} \right) \tag{21b}
\]

\[
F_{m \beta} = (p + q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_m(v_0') B_{\beta}(v_0) + \frac{c_{2}^{2}}{4} I_{m\beta}^{AB} \right) + dc \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_m(v_0') B_{\beta}(v_0) + \frac{c_{2}^{2}}{4} I_{m\beta}^{AB} \right) + (p - q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_{\beta}(v_0') B_{m}(v_0) + \frac{c_{2}^{2}}{4} I_{m\beta}^{AB} \right) \tag{21c}
\]

\[
G_{m \beta} = (p + q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} B_m(v_0') B_{\beta}(v_0) \exp\left( \frac{-2 \tau'}{v_0} \right) + \frac{c_{2}^{2}}{4} I_{m\beta}^{BBE} \right) + dc \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_m(v_0') B_{\beta}(v_0) \exp\left( \frac{-2 \tau'}{v_0} \right) + \frac{c_{2}^{2}}{4} I_{m\beta}^{ABE} \right) + dc \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} B_m(v_0') A_{\beta}(v_0') \exp\left( \frac{2 \tau'}{v_0} \right) + \frac{c_{2}^{2}}{4} I_{m\beta}^{ABE} \right) + (p - q) \left( \frac{c_{2}^{2} v_0^{2}}{4 N'(v_0')} A_{\beta}(v_0') A_m(v_0) \exp\left( \frac{2 \tau'}{v_0} \right) + \frac{c_{2}^{2}}{4} I_{m\beta}^{ABE} \right) \tag{21d}
\]
When we calculate the expansion coefficients \( a_i \) and \( b_i \) from Eqs. (20), we can compute the albedo and transmission factor for the slab albedo problem using the following equations:

\[
A^* = (\beta + 2) \int_0^1 \Psi(-\tau, -\mu) \mu \, d\mu = (\beta + 2) \sum_{\ell=0}^N a_{\ell} \left( \frac{\ell}{\ell+2} \right)
\]

\[
B^* = (\beta + 2) \int_0^1 \Psi(\tau, \mu) \mu \, d\mu = (\beta + 2) \sum_{\ell=0}^N b_{\ell} \left( \frac{\ell}{\ell+2} \right)
\]

3. CONCLUSIONS

In this paper, we have considered the slab-albedo problem with extremely anisotropic scattering. The analytical expressions of exit distributions in \( F_N \) method are used for each problem. The numerical results are obtained from the exit distributions which are multiplied by \( \mu^{m+1} \) \( (m=0, 1, 2...) \) and integrated over \( \mu \in [0,1] \), [8]. This is a different numerical calculation from \( F_N \) method. In Tables (1-2) the albedo and transmission factors of a slab are calculated for different values of \( \tau' \), \( \alpha \), \( \beta \) and \( \tau' \). Table (1) is compared with Ref.[14]. The numerical values of both methods converge identically. Table (1) is compared with Ref.[15]. In Ref.[15] the numerical results were convergence in \( F_{15} \).

Table 1. The albedo and transmission factor for different values of \( \tau' \) and \( \beta \)

\[
\begin{array}{cccc|cccc|cccc|cccc}
\hline
& \multicolumn{4}{c|}{\text{Forward scattering}} & \multicolumn{4}{c|}{\text{Backward scattering}} & \multicolumn{4}{c}{\text{Forward scattering}} & \multicolumn{4}{c}{\text{Backward scattering}} \\
N & \tau' = 1 & \tau' = 4 & \tau' = 1 & \tau' = 4 & \tau' = 1 & \tau' = 4 & \tau' = 1 & \tau' = 4 \\
0 & 0.27940 & 0.41575 & 0.34196 & 0.40660 & 0.45111 & 0.37999 & 0.53361 & 0.37623 \\
1 & 0.28017 & 0.41625 & 0.34112 & 0.40679 & 0.44764 & 0.35236 & 0.53206 & 0.38222 \\
2 & 0.28015 & 0.41625 & 0.34109 & 0.40678 & 0.44759 & 0.35236 & 0.53202 & 0.38190 \\
3 & 0.28015 & 0.41625 & 0.34109 & 0.40678 & 0.44759 & 0.35236 & 0.53202 & 0.38190 \\
4 & 0.28015 & 0.41624 & 0.34109 & 0.40678 & 0.44759 & 0.35236 & 0.53202 & 0.38190 \\
10 & 0.28015 & 0.41624 & 0.34109 & 0.40678 & 0.44759 & 0.35236 & 0.53202 & 0.38190 \\
Ref. (14) & 0.28015 & 0.41625 & - & - & 0.44759 & 0.35236 & - & - \\
C_{2} & & & & & & & & \\
\hline
\end{array}
\]

\[
\begin{array}{cccc|cccc|cccc|cccc}
\hline
& \multicolumn{4}{c|}{\text{Forward scattering}} & \multicolumn{4}{c|}{\text{Backward scattering}} & \multicolumn{4}{c}{\text{Forward scattering}} & \multicolumn{4}{c}{\text{Backward scattering}} \\
N & \beta = 1 & \beta = 4 & \beta = 1 & \beta = 4 & \beta = 1 & \beta = 4 & \beta = 1 & \beta = 4 \\
0 & 0.25693 & 0.45023 & 0.25349 & 0.49129 & 0.4205 & 0.38109 & 0.39556 & 0.42009 \\
1 & 0.25878 & 0.45173 & 0.25356 & 0.49462 & 0.42445 & 0.38653 & 0.39804 & 0.42853 \\
2 & 0.25869 & 0.45162 & 0.25302 & 0.49415 & 0.42443 & 0.38653 & 0.39750 & 0.42762 \\
3 & 0.25869 & 0.45162 & 0.25304 & 0.49420 & 0.42443 & 0.38653 & 0.39758 & 0.42771 \\
4 & 0.25869 & 0.45162 & 0.25304 & 0.49420 & 0.42443 & 0.38653 & 0.39758 & 0.42771 \\
10 & 0.25869 & 0.45162 & 0.25304 & 0.49420 & 0.42443 & 0.38653 & 0.39758 & 0.42771 \\
Ref. (14) & 0.25869 & 0.45162 & - & - & 0.42443 & 0.38635 & - & - \\
C_{2} & & & & & & & & \\
\hline
\end{array}
\]
Table 2. The albedo and transmission factor for $c = 0.3, \tau = 5, \beta = 0$ and different values of $\alpha$

<table>
<thead>
<tr>
<th>N</th>
<th>$\alpha = 0.2$</th>
<th>$\alpha = 0.4$</th>
<th>$\alpha = 0.6$</th>
<th>$\alpha = 0.8$</th>
<th>$\alpha = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.06136</td>
<td>0.00379</td>
<td>0.04747</td>
<td>0.00478</td>
<td>0.03269</td>
</tr>
<tr>
<td>1</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
</tr>
<tr>
<td>2</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
</tr>
<tr>
<td>3</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
</tr>
<tr>
<td>4</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
</tr>
<tr>
<td>10</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
</tr>
<tr>
<td>20</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ref(15)</th>
<th>$A^*$</th>
<th>$B^*$</th>
<th>$A^*$</th>
<th>$B^*$</th>
<th>$A^*$</th>
<th>$B^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^*$</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
<td>0.00605</td>
</tr>
<tr>
<td>$B^*$</td>
<td>0.06135</td>
<td>0.00379</td>
<td>0.04746</td>
<td>0.00478</td>
<td>0.03269</td>
<td>0.00605</td>
</tr>
</tbody>
</table>

4. REFERENCES


